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Journal of Approximation Theory 134 (2005) 137–170

Journal of Approximation Theory

www.elsevier.com/locate/jat

Riemann–Hilbert analysis for Jacobi polynomials orthogonal on a single contour

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Received 12 October 2004; accepted 16 February 2005

Communicated by Arno B.J. Kuijlaars
Available online 8 April 2005

Abstract

Classical Jacobi polynomials $P_n^{(\alpha,\beta)}$, with $\alpha, \beta > -1$, have a number of well-known properties, in particular the location of their zeros in the open interval $(-1, 1)$. This property is no longer valid for other values of the parameters; in general, zeros are complex. In this paper we study the strong asymptotics of Jacobi polynomials where the real parameters α_n, β_n depend on n in such a way that

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = A, \quad \lim_{n\to\infty} \frac{\beta_n}{n} = B,$$

with $A, B \in \mathbb{R}$. We restrict our attention to the case where the limits A, B are not both positive and take values outside of the triangle bounded by the straight lines $A=0$, $B=0$ and $A+B+2=0$. As a corollary, we show that in the limit the zeros distribute along certain curves that constitute trajectories of a quadratic differential.

The non-hermitian orthogonality relations for Jacobi polynomials with varying parameters lie in the core of our approach; in the cases we consider, these relations hold on a single contour of the complex plane. The asymptotic analysis is performed using the Deift–Zhou steepest descent method based on the Riemann–Hilbert reformulation of Jacobi polynomials.
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Keywords: Asymptotics; Non-hermitian orthogonality; Steepest descent method; Riemann–Hilbert characterization

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doi:10.1016/j.jat.2005.02.004

1. Introduction

Jacobi polynomials $P_n^{(\alpha,\beta)}$ are given explicitly by

$$P_n^{(\alpha,\beta)}(z) = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (z-1)^k (z+1)^{n-k}, \tag{1.1}$$

or, equivalently, by the Rodrigues formula [35, Chapter IV]

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{2^n n!} (z-1)^{-\alpha}(z+1)^{-\beta} \left(\frac{d}{dz}\right)^n \left[(z-1)^{n+\alpha}(z+1)^{n+\beta}\right]. \tag{1.2}$$

In the classical situation ($\alpha, \beta > -1$) the Jacobi polynomials are orthogonal in $[-1, 1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ and, consequently, their zeros are simple and located in $(-1, 1)$.

Expressions (1.1) and (1.2) show that the definition of $P_n^{(\alpha,\beta)}$ may be extended to arbitrary $\alpha, \beta \in \mathbb{R}$ (or even $\mathbb{C}$); but some properties of the classical polynomials, in particular the location and simplicity of the zeros, are no longer valid. In fact, $P_n^{(\alpha,\beta)}$ may have a multiple zero at $z = 1$ if $\alpha \in \{-1, \ldots, -n\}$, at $z = -1$ if $\beta \in \{-1, \ldots, -n\}$ or at $z = \infty$ (which means a degree reduction) if $n + \alpha + \beta \in \{-1, \ldots, -n\}$.

More precisely, for $k \in \{1, \ldots, n\}$, we have (see [35, formula (4.22.2)]),

$$P_n^{(-k,\beta)}(z) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+1-k)} \frac{(n-k)!}{n!} \left(\frac{z-1}{2}\right)^k P_{n-k}^{(k,\beta)}(z). \tag{1.3}$$

This implies in particular that $P_n^{(-k,\beta)}(z) \equiv 0$ if additionally $\max\left\{k, -\beta\right\} \leqslant n \leqslant k-\beta-1$. Analogous relations hold for $P_n^{(\alpha,-l)}$ when $l \in \{1, \ldots, n\}$. Thus, when both $k, l \in \mathbb{N}$ and $k + l \leqslant n$, we have

$$P_n^{(-k,-l)}(z) = 2^{-k-l} (z-1)^k (z+1)^l P_{n-k-l}^{(k,l)}(z). \tag{1.4}$$

Furthermore, when $n + \alpha + \beta = -k \in \{-1, \ldots, -n\}$,

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha)} \frac{(k-1)!}{n!} P_{k-1}^{(\alpha,\beta)}(z), \tag{1.5}$$

see [35, Eq. (4.22.3)]; we refer the reader to [35, §4.22] for a more detailed discussion. Taking into account formulas (1.3)–(1.5) we exclude these special integer parameters from our further analysis.

Jacobi polynomials $P_n^{(\alpha,\beta)}$ with parameters $\alpha, \beta \in \mathbb{R}$ (and in general, depending on the degree n) appear naturally as polynomial solutions of hypergeometric differential equations, or in the expressions of the wave functions of many classical systems in quantum mechanics (see e.g. [2]), or even in the explicit evaluation of integrals of rational functions [4].

In this paper we study the asymptotic behavior of Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}$, where the parameters α_n, β_n depend on the degree n in such a way that

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = A, \quad \lim_{n\to\infty} \frac{\beta_n}{n} = B \tag{1.6}$$

and

$$A < 0 < B, \quad A + B > -1, \quad A \neq -1. \tag{1.7}$$

In [27], the authors considered different regions of the (A, B)-plane, corresponding to different cases in the asymptotic study of Jacobi polynomials with varying parameters satisfying (1.6). The symmetry relations (see [35, Chapter IV])

$$P_n^{(\alpha,\beta)}(z) = (-1)^n P_n^{(\beta,\alpha)}(z) \tag{1.8}$$

and

$$P_n^{(\alpha,\beta)}(z) = \left(\frac{1-z}{2}\right)^n P_n^{(\alpha',\beta)}\left(\frac{z+3}{z-1}\right), \tag{1.9}$$

where $\alpha' = -2n - \alpha - \beta - 1$, allow to restrict our study to the following cases, from which all the others can be obtained:

$$A, B > 0, \tag{C.1}$$

$$A < -1 \quad \text{and} \quad A + B > -1, \tag{C.2}$$

$$-1 < A < 0 \quad \text{and} \quad B > 0, \tag{C.3}$$

$$A + B > -1, \quad \text{and} \quad A, B < 0, \tag{C.4}$$

$$A + B < -1 \quad \text{and} \quad A, B > -1 \tag{C.5}$$

(see Fig. 1, which appeared first in [27], where equivalent regions under those transformations are shown).

Case C.1 is classical and has been widely studied (see [5,6,11,20,26,29]). The asymptotic results therein are based on either the well-known orthogonality conditions satisfied by the Jacobi polynomials on $[-1, 1]$, or on their integral representation.

However, until very recently, strong asymptotics of $P_n^{(\alpha_n,\beta_n)}$, when α_n, β_n take *arbitrary real* values and limits (1.6) exist, has not been established. In this case the orthogonality conditions were unknown and the complex saddle points make the application of the classical steepest descent method to the integral representation of Jacobi polynomials practically unfeasible.

A non-hermitian orthogonality satisfied by Jacobi polynomials in case C.2 has been observed in [27]; this fact was used there to establish the asymptotic zero distribution using a potential theory approach.

Recently in [25] a whole spectrum of orthogonality conditions for Jacobi polynomials with arbitrary real parameters has been established. In particular, we can find examples of orthogonality on a contour or arc of the complex plane, an incomplete or quasi-orthogonality, or even multiple or Hermite–Padé orthogonality conditions. The classification of the cases depends on the number of inequalities $-1 < A < 0$, $-1 < B < 0$, $-2 < A + B < -1$ that are satisfied. In particular, cases C.3–C.5 correspond to combinations of parameters A

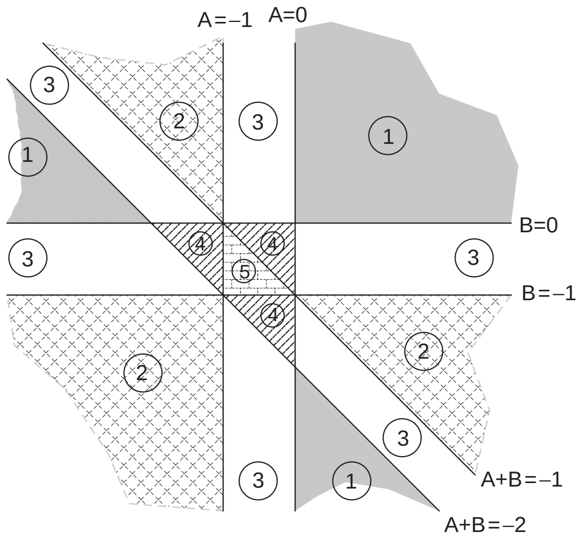

Fig. 1. The five different cases in the classification of Jacobi polynomials with varying parameters according to [27].

and B such that exactly one, exactly two, or exactly three, respectively, of the inequalities are satisfied (cf. Fig. 1).

Nevertheless, the method used in [27] cannot be immediately extended to the rest of the cases. One of the essential assumptions there is a non-hermitian orthogonality of the polynomials on a single contour, on which the support of a certain equilibrium measure has a connected complement.

Due to this reason, in [22] the steepest descent method of Deift and Zhou [10], based on a matrix Riemann–Hilbert problem, was used to establish the strong uniform asymptotics of the Jacobi polynomials with parameters satisfying conditions C.5. In this paper, we use several results and ideas from there.

The aim of the present article is to extend this analysis to sequences of Jacobi polynomials with varying parameters corresponding to cases C.2 and C.3. Note that along with case C.1, these are the only situations when a full system of orthogonality relations on a single contour in $\mathbb{C}$ exists.

We also remark that a similar study, but for Laguerre polynomials with varying parameters, has been carried out in [23,24,28].

The paper is organized as follows. In Section 2, the main results about strong and weak zero asymptotics are formulated, along with some preliminary definitions and lemmas which are proved in Section 3. In Section 4, a full set of orthogonality relations on a single contour allows to pose a Riemann–Hilbert problem and to apply the Deift and Zhou's steepest descent method (see [10], and also [3], where this method was applied on trajectories of a quadratic differentials for the first time). This technique allows us to transform the original Riemann–Hilbert problem in order to obtain strong asymptotics of its solution. Finally, the last section is devoted to the proofs of the main results.

2. Main results

2.1. Basic definitions

Let us denote by $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $\mathbb{C}^- = \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$. For A, B satisfying (1.7), we define the points

$$\zeta_{1,2} = \frac{B^2 - A^2 \pm 4\sqrt{(A+1)(B+1)(A+B+1)}}{(A+B+2)^2} \tag{2.1}$$

(for a motivation of this definition see Section 5.3). We will use the following convention: for (A, B) such that $A < -1 < A+B$ (case C.2), $\zeta_1 \in \mathbb{C}^+$ and $\zeta_2 = \overline{\zeta_1}$; for $-1 < A < 0 < B$ (case C.3), we agree that $-1 < \zeta_1 < \zeta_2 < 1$.

With these $\zeta_{1,2}$, consider the set

$$\mathcal{N} \stackrel{\text{def}}{=} \left\{ z \in \mathbb{C} : \operatorname{Re} \int_{\zeta_1}^{z} \frac{((t-\zeta_1)(t-\zeta_2))^{1/2}}{t^2-1}\, dt = 0 \right\}, \tag{2.2}$$

where we continue the integrand analytically along the path of integration. Obviously, the set does not depend on the branch of the square root. In fact, it coincides with the union of the critical trajectories of the quadratic differential

$$-\frac{(z-\zeta_1)(z-\zeta_2)}{(z^2-1)^2}\, dz^2, \tag{2.3}$$

or more precisely, with their projection on $\mathbb{C}$. Taking into account the local structure of trajectories of quadratic differentials (see e.g. [31] or [34]), we can prove the following (see Fig. 2):

Lemma 1. *If parameters A, B satisfy condition* (1.7), *then for* $\zeta_{1,2}$ *defined in* (2.1) *the quadratic differential is regular. In other words, all its critical trajectories are finite and have the following global structure* (*see Fig.* 2):

- *For* (A, B) *such that* $A < -1 < A + B$ (*case* C.2), $\mathcal{N}$ *consists of three arcs which connect* $\zeta_{1,2}$ *and intersect the real line in exactly one point, in such a way that each of the intervals* $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$ *is cut by one of these arcs.*
- *For* (A, B) *such that* $-1 < A < 0 < B$ (*case* C.3), $\mathcal{N}$ *consists of three arcs*; *one of them is the real interval* $[\zeta_1, \zeta_2]$ *and the other two are Jordan contours, passing through* $z = \zeta_1$ (*respect.*, $z = \zeta_2$) *and enclosing* $z = -1$ (*respect.*, $z = 1$).

Now we define some relevant curves. We denote by Γ the rightmost curve from $\mathcal{N}$. For case C.2, Γ consists of an arc connecting $\zeta_{1,2}$ and crossing once the interval $(1, +\infty)$, and for case C.3, it is a closed contour passing through $z = \zeta_2$ and surrounding $z = 1$. For case C.2 we also consider the orthogonal trajectories $\mathcal{N}^\perp$ (defined by replacing Re in (2.2) by Im). As in Lemma 1, it is easy to prove that their global structure is as appears in Fig. 2, left (dashed lines). We denote by γ^+ the arc of $\mathcal{N}^\perp$ joining ζ_1 and -1, and $\gamma^- = \overline{\gamma^+}$.

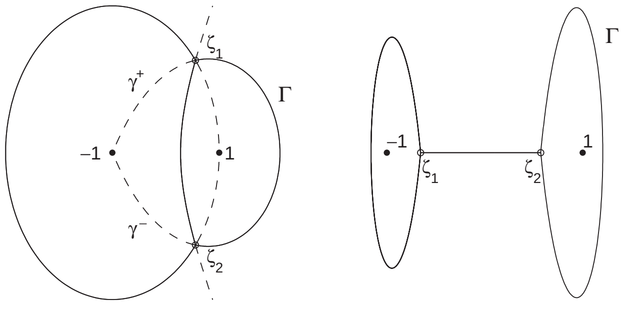

Fig. 2. Typical structure of the set $\mathcal{N}$ for cases C.2 (left) and C.3. Dashed lines on the left are orthogonal critical trajectories.

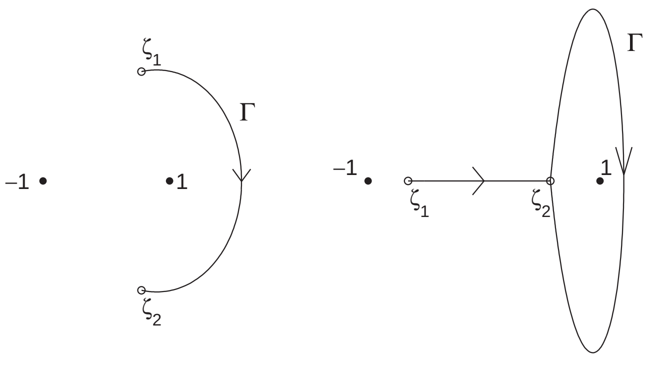

Fig. 3. Contours Σ for cases C.2 (left) and C.3.

Finally, we define the set Σ as the smallest connected subset of $\mathcal{N}$ containing $\zeta_{1,2}$ and Γ. Namely,

$$\Sigma \stackrel{\text{def}}{=} \begin{cases} \Gamma & \text{if } A < -1 < A + B, \ \text{(case C.2)}, \\ \Gamma \cup [\zeta_1, \zeta_2] & \text{if } -1 < A < 0 < B, \ \text{(case C.3)}. \end{cases} \tag{2.4}$$

As we see, in case C.2 the set Σ is made of one critical trajectory of the quadratic differential (2.3), while in case C.3 it is made of two. In both cases Σ is oriented from ζ_1 to ζ_2, and, in case C.3, clockwise, in such a way that $(1, +\infty)$ is cut from the upper to the lower half-plane (see Fig. 3). For any function *f* analytic and single-valued in a neighborhood of Σ, this selection of the orientation induces two boundary values of *f* on Σ that we denote by f_+ and f_- depending if we approach Σ from the left or from the right, respectively. On the sequel, we shall make use of the concept of the polynomial convex hull of Σ, which is denoted by $\mathrm{Pc}(\Sigma)$. In case C.2, $\mathrm{Pc}(\Sigma) = \Gamma$, so that $\mathrm{Int}(\mathrm{Pc}(\Sigma)) = \emptyset$, where by $\mathrm{Int}(e)$ we denote the set of inner points of *e*. Analogously, in case C.3, $\mathrm{Pc}(\Sigma)$ is the union of Σ and of the closure of the bounded component of its complement, given by $\mathrm{Int}(\mathrm{Pc}(\Sigma))$.

Next, we define some functions that will play a role in what follows. We denote

$$R(z) \stackrel{\text{def}}{=} \sqrt{(z-\zeta_1)(z-\zeta_2)}.$$

It is a multi-valued and analytic function in $\mathbb{C}$, and we select its single-valued branch in a plane cut from ζ_1 to ζ_2 by imposing that

$$\lim_{z\to\infty} \frac{R(z)}{z} = 1.$$

This allows us to define the (a priori complex) measure

$$d\mu(z) = \frac{A+B+2}{2\pi i}\,\frac{R_+(z)}{1-z^2}\,dz, \quad z \in \Sigma, \tag{2.5}$$

with Σ defined in (2.4) and oriented as explained. By Lemma 2 below, μ is a unit positive measure on Σ. Using Cauchy's Theorem it is easy to find an analytic expression for its Cauchy transform:

$$\widehat{\mu}(z) \stackrel{\text{def}}{=} \int \frac{d\mu(t)}{z-t} = \frac{A+B+2}{2}\,\frac{R(z)}{z^2-1} - \frac{A/2}{z-1} - \frac{B/2}{z+1}, \quad z \in \mathbb{C}\setminus \mathrm{Pc}(\Sigma), \tag{2.6}$$

additionally, in case C.3,

$$\widehat{\mu}(z) = -\frac{A+B+2}{2}\,\frac{R(z)}{z^2-1} - \frac{A/2}{z-1} - \frac{B/2}{z+1}, \quad z \in \mathrm{Int}(\mathrm{Pc}(\Sigma)). \tag{2.7}$$

Now, let us define in $\mathbb{C}\setminus\Sigma$ a function which plays a key role in the description of the strong asymptotics of Jacobi polynomials,

$$G(z) \stackrel{\text{def}}{=} \exp\left(\int_{\zeta_2}^{z} \widehat{\mu}(t)\,dt\right). \tag{2.8}$$

We normalize G by imposing that

$$\lim_{z\to\zeta_2} G(z) = 1, \tag{2.9}$$

where the limit is taken with z approaching ζ_2 from $\mathbb{C}\setminus\Gamma$ (in case C.2) or from $\mathbb{C}^+\setminus\Sigma$ (in case C.3). Observe that since $\widehat{\mu}$ is the Cauchy transform of a unit measure on Σ, function G is analytic and single-valued in $\mathbb{C}\setminus\Sigma$ in both cases considered. Taking into account (2.6) and (2.7), we see that there exists

$$\kappa \stackrel{\text{def}}{=} \lim_{z\to\infty} \frac{G(z)}{z}. \tag{2.10}$$

In addition, let

$$\begin{aligned} w(z) = w(z; A, B) &\stackrel{\text{def}}{=} c(z-1)^{A/2}(z+1)^{B/2} \\ &= \exp\left(\int_{\zeta_2}^{z}\left(\frac{A/2}{t-1} + \frac{B/2}{t+1}\right)dt\right), \end{aligned} \tag{2.11}$$

which is a multi-valued analytic function in $\mathbb{C} \setminus \{\pm 1\}$. In what follows, we fix its single-valued analytic branch in $\mathbb{C} \setminus (-\infty, 1]$, by choosing the constant c (or the path of integration) in such a way that

$$\lim_{z \to \zeta_2} w(z) = 1, \tag{2.12}$$

where again the limit is taken with z approaching ζ_2 from $\mathbb{C} \setminus \Gamma$ (in case C.2) or from $\mathbb{C}^+ \setminus \Sigma$ (in case C.3).

The last ingredient for the asymptotics is given by the functions

$$N_{11}(z) \stackrel{\text{def}}{=} \frac{a(z) + a(z)^{-1}}{2} \quad \text{and} \quad N_{12}(z) \stackrel{\text{def}}{=} \frac{a(z) - a(z)^{-1}}{2i} \tag{2.13}$$

(this notation is chosen because they will be entries of a certain matrix N, see (4.15)), where

$$a(z) \stackrel{\text{def}}{=} \left(\frac{z - \zeta_2}{z - \zeta_1} \right)^{\frac{1}{4}} \tag{2.14}$$

is defined in $\mathbb{C} \setminus \Gamma$ for $A < -1 < A + B$ (case C.2), and in $\mathbb{C} \setminus [\zeta_1, \zeta_2]$ for $-1 < A < 0 < B$ (case C.3). We select the branch of a imposing the normalization condition

$$\lim_{z \to \infty} a(z) = 1.$$

Then, $N_{11}(z) \to 1$ and $N_{12}(z) \to 0$ as $z \to \infty$.

2.2. *Strong asymptotics*

First, we consider the strong asymptotics for Jacobi polynomials with varying parameters satisfying (1.6) and (1.7) with z away from Σ.

Theorem 1. *Let* (A, B) *satisfy* (1.7). *Then, for* $n \to \infty$, *the monic Jacobi polynomials* $p_n = \widehat{P}_n^{(An, Bn)}$ *have the following asymptotic behavior*:

$$p_n(z) = \left(\frac{G(z)}{\kappa} \right)^n N_{11}(z) \left(1 + O\left(\frac{1}{n} \right) \right), \tag{2.15}$$

locally uniformly in $\mathbb{C} \setminus \mathrm{Pc}(\Sigma)$, *where constant* κ *was defined in* (2.10).

Furthermore, in the bounded component of $\mathbb{C} \setminus \Sigma$,

$$p_n(z) = \frac{1}{\kappa^n} \left(\left(G(z) w^2(z) \right)^{-n} N_{11}(z) \left(1 + O\left(\frac{1}{n} \right) \right) \right.$$
$$\left. + 2i e^{-An\pi i} \sin(A\pi n)\, G^n(z) N_{12}(z) \left(1 + O\left(\frac{1}{n} \right) \right) \right). \tag{2.16}$$

In particular, this theorem shows that zeros of $P_n^{(An, Bn)}$ do not accumulate in $\mathbb{C} \setminus \mathrm{Pc}(\Sigma)$.

Next, we describe the asymptotics on Σ, but away from the branch points $\zeta_{1,2}$:

Theorem 2. *Let* (A, B) *satisfy* (1.7). *Then, for* $n \to \infty$, *the monic Jacobi polynomials* $p_n = \widehat{P}_n^{(An,Bn)}$ *have the following asymptotic behavior for z away from* $\zeta_{1,2}$:

(a) *In case* C.2, *on the* "$\pm$"*-side of* Γ

$$p_n(z) = \frac{1}{\kappa^n} \left(G^n(z) N_{11}(z) \left(1 + O\left(\frac{1}{n}\right)\right) \right.$$
$$\left. \pm \left(G(z) w^2(z)\right)^{-n} N_{12}(z) \left(1 + O\left(\frac{1}{n}\right)\right)\right). \tag{2.17}$$

(b) *In case* C.3, *on the* "$-$"*-side of* Γ *formula* (2.16) *is still valid, while on the* "$+$"*-side of* Γ,

$$p_n(z) = \frac{1}{\kappa^n} \left(G^n(z) N_{11}(z) \left(1 + O\left(\frac{1}{n}\right)\right) \right.$$
$$\left. + 2ie^{-An\pi i} \sin(A\pi n) \left(G(z) w^2(z)\right)^{-n} N_{12}(z) \left(1 + O\left(\frac{1}{n}\right)\right)\right). \tag{2.18}$$

(c) *In case* C.3, *on the* "$+$"*-side of* (ζ_1, ζ_2),

$$p_n(z) = \frac{1}{\kappa^n} \left(G^n(z) N_{11}(z) \left(1 + O\left(\frac{1}{n}\right)\right) \right.$$
$$\left. + \left(G(z) w^2(z)\right)^{-n} N_{12}(z) \left(1 + O\left(\frac{1}{n}\right)\right)\right), \tag{2.19}$$

while on the "$-$"*-side of* (ζ_1, ζ_2),

$$p_n(z) = \frac{1}{\kappa^n} \left(G^n(z) N_{11}(z) \left(1 + O\left(\frac{1}{n}\right)\right) \right.$$
$$\left. - e^{-2A\pi i n} \left(G(z) w^2(z)\right)^{-n} N_{12}(z) \left(1 + O\left(\frac{1}{n}\right)\right)\right). \tag{2.20}$$

Remark 1. All these asymptotic expressions match on the boundaries of the overlapping domains and on the respective regions. For instance, it will be shown that in a small neighborhood of every point of Σ (distinct from $\zeta_{1,2}$), $|G(z)w(z)| > 1$ for $z \notin \Sigma$. Hence, the first term in (2.17)–(2.20) is dominant, and away from Σ they reduce to (2.15).

Furthermore, in case C.3, if *An* are not exponentially close to integers (in the sense that will be made more precise below), the second term in (2.16) dominates, and we may write

$$p_n(z) = \left(\frac{G(z)}{\kappa}\right)^n 2ie^{-An\pi i} \sin(A\pi n) N_{12}(z) \left(1 + O\left(\frac{1}{n}\right)\right).$$

The asymptotic formulas above are no longer valid close to the branch points $\zeta_{1,2}$. As it usually happens in a neighborhood of the "soft ends" of the support of an equilibrium measure, asymptotics is described in terms of the Airy function $\mathrm{Ai}(z)$ and its derivative. We give explicit formulas only for the rightmost (according to the orientation of Σ) branch point ζ_2, which is in a certain sense, the "interesting" one. Clearly, the analysis at the other point is similar.

In order to formulate our result in a more compact form it is convenient to introduce the function

$$\phi(z) \stackrel{\text{def}}{=} \frac{A+B+2}{2} \int_{\zeta_2}^{z} \frac{R(t)}{1-t^2}\, dt. \tag{2.21}$$

Consider case C.2. Given a sufficiently small $\varepsilon > 0$, and a neighborhood $\Delta_\varepsilon(\zeta_2) \stackrel{\text{def}}{=} \{z \in C : |z-\zeta_2| < \varepsilon\}$, it is a single-valued analytic function in $\Delta_\varepsilon(\zeta_2) \setminus \Gamma$. Furthermore, taking into account the local behavior of *R*, function

$$f(z) \stackrel{\text{def}}{=} \tfrac{3}{2}\, \left(\phi(z)\right)^{2/3} \tag{2.22}$$

can be extended as single-valued to the whole $\Delta_\varepsilon(\zeta_2)$. Here the 2/3rd power is chosen such that $f(z) > 0$ along γ^-.

Theorem 3. *Let* (A, B) *such that* $A < -1 < A+B$ (*case* C.2). *Then, there exists* $\varepsilon > 0$ *such that if* $|z-\zeta_2| < \varepsilon$, *we have that the monic Jacobi polynomials* $p_n = \widehat{P}_n^{(An,Bn)}$ *satisfy*

$$p_n(z) = \frac{\sqrt{\pi}}{\kappa^n w^n(z)} \left(\frac{n^{1/6}\, f^{1/4}(z)}{a(z)} \operatorname{Ai}(n^{2/3} f(z)) \left(1 + O\left(\frac{1}{n}\right)\right) \right.$$
$$\left. - \frac{a(z)}{n^{1/6}\, f^{1/4}(z)} \operatorname{Ai}'(n^{2/3} f(z)) \left(1 + O\left(\frac{1}{n}\right)\right)\right),$$

where $a(z)$ *is defined in* (2.14), *and we take* $f^{1/4}(z) > 0$ *along* γ^-.

Remark 2. Obviously, the asymptotic behavior near ζ_1 in this case is completely symmetric to ζ_2 with respect to $\mathbb{R}$.

Consider case C.3. For a sufficiently small $0 < \varepsilon < 1 - \zeta_2$, ϕ is single-valued and analytic in $\Delta_\varepsilon(\zeta_2) \setminus (\zeta_1, \zeta_2)$. Function *f*, defined again by formula (2.22), can be extended as a single-valued function to the whole $\Delta_\varepsilon(\zeta_2)$, with the 2/3rd power chosen such that $f(z) > 0$ along $(\zeta_2, 1)$.

Theorem 4. *Let* (A, B) *such that* $-1 < A < 0 < B$ (*case* C.3). *Then, there exists* $\varepsilon > 0$ *such that if* $|z-\zeta_2| < \varepsilon$, *we have that the monic Jacobi polynomials* $p_n = \widehat{P}_n^{(An,Bn)}$ *satisfy*

$$p_n(z) = \frac{\sqrt{\pi}}{\kappa^n w^n(z)} \left(\frac{n^{1/6}\, f^{1/4}(z)}{a(z)} \mathcal{A}(n^{2/3} f(z)) \left(1 + O\left(\frac{1}{n}\right)\right) \right.$$
$$\left. - \frac{a(z)}{n^{1/6}\, f^{1/4}(z)} \mathcal{A}'(n^{2/3} f(z)) \left(1 + O\left(\frac{1}{n}\right)\right)\right), \tag{2.23}$$

where $a(z)$ *is defined in* (2.14),

$$\mathcal{A}(t) = \mathcal{A}(t; A, n) \stackrel{\text{def}}{=} e^{-A\pi i n} \operatorname{Ai}(t) + 2i e^{\pi i/3} \sin(A\pi n) \operatorname{Ai}\left(e^{4\pi i/3}\, t\right),$$

and we take $f^{1/4}(z) > 0$ *along* $(\zeta_2, 1)$.

Remark 3. Formulas stated in Theorems 1–4 are locally uniformly continuous both on the z and (A, B) planes, which allows to extend them to the general case of $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying (1.6) and (1.7).

2.3. Weak zero asymptotics

As a corollary of the asymptotic formulas stated in the previous section we can obtain the distribution of the zeros of the sequence of polynomials $P_n^{(\alpha_n,\beta_n)}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy (1.6) and (1.7). By "weak zero asymptotics" we understand here the limit (in the weak-* sense) of the normalized zero counting measures associated with $P_n^{(\alpha_n,\beta_n)}$.

The measure μ introduced in (2.5) will be all we need for the description of the asymptotic behavior of the zeros in case C.2. However, region C.3 comprises the pathological cases given by (1.3). By continuity, we may expect here a variety of limit behaviors. In fact, in order to characterize completely the weak zero asymptotics of Jacobi polynomials in the case C.3 we need to use a 1-parametric family of measures including (2.5). Namely, in case C.3, we must consider the sets

$$\begin{aligned}\mathcal{N}_r = \mathcal{N}_r^{(A,B)} &\stackrel{\text{def}}{=} \left\{ z \in \mathbb{C} :\ |G(z)w(z)| = e^{r/2} \right\} \\ &= \left\{ z \in \mathbb{C} :\ \operatorname{Re} \int_{\zeta_2}^{z} \frac{R(t)}{t^2-1}\, dt = \frac{r}{A+B+2} \right\},\end{aligned}$$

for $r \geqslant 0$. They also consist of trajectories of the quadratic differential (2.3), and $\mathcal{N}_0 = \mathcal{N}$. Now, we define Γ_r as the rightmost curve in $\mathcal{N}_r$ or, what is the same, the part of $\mathcal{N}_r$ which is entirely contained in the half-plane $\{z \in \mathbb{C} :\ \operatorname{Re} z \geqslant \zeta_2\}$. It is easy to check that for $r > 0$ the level curve Γ_r is a closed contour inside $\Gamma = \Gamma_0$ surrounding the point $z = 1$ (see Fig. 4).

For each $r \in [0, \infty)$ we define the absolutely continuous measure

$$d\mu_r(z) \stackrel{\text{def}}{=} \frac{A+B+2}{2\pi i} \frac{R_+(z)}{1-z^2}\, dz, \quad z \in (\zeta_1, \zeta_2) \cup \Gamma_r, \tag{2.24}$$

and, for $r = \infty$, the measure

$$d\mu_\infty(z) \stackrel{\text{def}}{=} -A\delta_1 + \frac{A+B+2}{2\pi} \frac{\sqrt{(z-\zeta_1)(\zeta_2-z)}}{1-z^2} \chi_{[\zeta_1,\zeta_2]}\, dz, \tag{2.25}$$

where $\chi_{[\zeta_1,\zeta_2]}$ is the characteristic function of the interval $[\zeta_1, \zeta_2]$, and δ_1 is the Dirac delta (unit mass point) at $z = 1$.

Lemma 2. *If* (A, B) *satisfy* (1.7), *then for* $r \geqslant 0$ *measures* μ_r *in* (2.24) and (2.25) (*and, in particular, measure* μ *in* (2.5)) *are unit positive measures. Moreover, for* (A, B) *satisfying* $-1 < A < 0 < B$ (*case* C.3), *we have that for* $0 \leqslant r \leqslant +\infty$,

$$\mu_r([\zeta_1, \zeta_2]) = 1 + A, \quad \mu_r(\Gamma_r) = -A, \tag{2.26}$$

where we consider $\Gamma_0 = \Gamma$ *and* $\Gamma_\infty = \{1\}$.

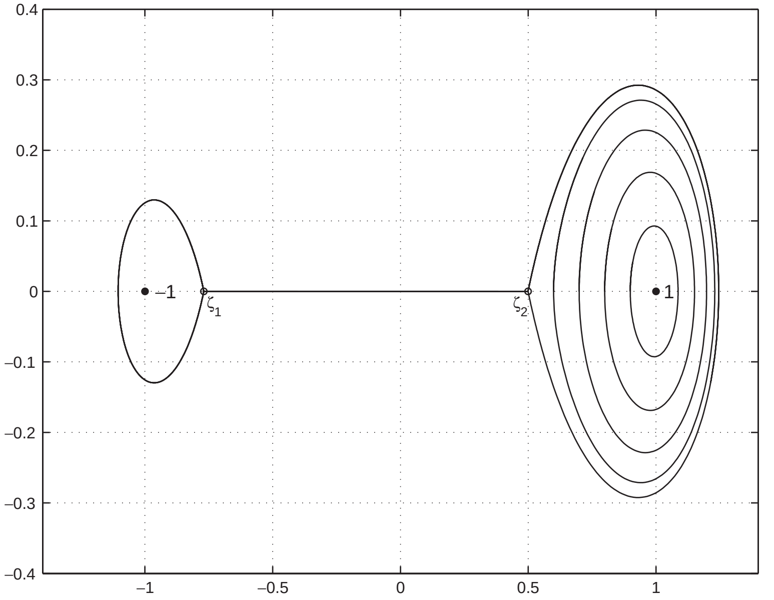

Fig. 4. Some trajectories of the quadratic differential (2.3), or equivalently, some level sets Γ_r, for the values $A = -0.8$ and $B = 0.5$.

Now we are ready to state the weak zero asymptotics for the Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}$. In case C.3, when $-1 < A < 0 < B$, we make an additional assumption: the sequence of parameters α_n satisfies that the following limit

$$\lim_{n\to\infty} [\mathrm{dist}(\alpha_n, \mathbb{Z})]^{1/n} = e^{-r}, \quad 0 \leqslant r \leqslant +\infty, \tag{2.27}$$

exists. Then, it holds:

Theorem 5. *Consider a sequence of Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}$, $n \in \mathbb{N}$, such that sequences $\{\alpha_n\}$, $\{\beta_n\}$ satisfy* (1.6) *and* (1.7). *Then*:

(i) *If (A, B) satisfy condition* C.2, *then the zeros of $P_n^{(\alpha_n,\beta_n)}$, $n \in \mathbb{N}$, accumulate on the arc Γ, and measure μ in* (2.5) *is the weak* limit of the corresponding normalized zero counting measures.*

(ii) *If (A, B) satisfy condition* C.3, *and* (2.27) *holds for some $0 \leqslant r \leqslant +\infty$, then the zeros of Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}$, $n \in \mathbb{N}$, accumulate on $[\zeta_1, \zeta_2] \cup \Gamma_r$, and measure μ_r defined above is the weak* limit of the normalized zero counting measures.*

As we said before, part (i) of Theorem 5, corresponding to case C.2, was established in [27] for parameters α_n, β_n varying according to (1.6) but with $A, B < -1$, which is a region of the (A, B)-plane equivalent to $A < -1 < A + B$ by means of transformations (1.8) and (1.9).

In the case C.3 the situation when $r = 0$, that is,

$$\lim_{n\to\infty} [\mathrm{dist}(\alpha_n, \mathbb{Z})]^{1/n} = 1,$$

is generic, because it takes place when parameters do not approach the integers exponentially fast. When the limit in (2.27) is smaller than one, i.e. $r > 0$, curve Γ is replaced by a level curve Γ_r, strictly contained inside the bounded component of the complement to Γ, and the support of the limit measure becomes disconnected. Finally, when parameters α_n, $n \in \mathbb{N}$, tend to integers faster than exponentially, the limit measure has a discrete part consisting of a Dirac mass at $z = 1$. We could have anticipated this phenomenon observing the coalescence of zeros given by (1.3).

Examples of zeros of Jacobi polynomials for cases C.2 and C.3 are represented in Fig. 5.

Remark 4. Case C.3 deals with situations when real zeros arise. When $-n < \alpha_n < 0$ and $\beta_n > -1$, $P_n^{(\alpha,\beta)}$ satisfy a quasi-orthogonality relation (see Theorem 6.1 in [25]) which ensures the existence of, at least, $n - [-\alpha_n]$ zeros in $(-1, 1)$. This lower bound of the number of zeros in $(-1, 1)$ is exact, according to the so-called Hilbert–Klein formulas [35, Theorem 6.72]. Since $\lim_{n\to\infty} \frac{n-[-\alpha_n]}{n} = 1 + A$, looking at (2.26) we see that the mass of the part of the asymptotic measure of zeros supported on $[\zeta_1, \zeta_2] \subset (-1, 1)$ agrees with the limit of the ratio of zeros placed in $(-1, 1)$, given by the Hilbert–Klein formulas.

Remark 5. At this point, it is natural to ask about what happens when $A = -1$ and $B > 0$, which is a transition case between C.2 and C.3. By (1.8) and (1.9), it describes also the situation when (A, B) belongs to any of the straight lines $A = -1$, $B = -1$ and $A + B = -1$, outside of the square $(A, B) \in [-1, 0] \times [-1, 0]$. Roughly speaking, in this case the endpoints $\zeta_{1,2}$ are confluent in a single point, say ζ, and zeros approach a simple closed contour emanating from this point (critical trajectory) and surrounding $z = 1$, or other closed trajectories strictly contained in the interior of this critical trajectory. In [27], the equation of this critical trajectory is conjectured. This conjecture is proved in [12,17] for the particular case where $\alpha_n = -n - 1$ and $\beta_n = kn + 1$, k being a fixed positive real number. This and the other transitions between cases C.1 and C.5 deserve a separated treatment.

3. Proof of the auxiliary lemmas

Proof of Lemma 1. This proof is based upon the local structure of the trajectories of quadratic differentials (see [31] or [34]). We restrict our attention to the case where $-1 < A < 0 < B$ (case C.3). The proof of the other case is similar (see also the proof of Lemma 2.1 in [22]).

First, we see that for $-1 < A < 0 < B$ the quadratic differential (2.3) possesses two simple zeros at $\zeta_{1,2}$. Thus, we know that three critical trajectories emanate from $\zeta_{1,2}$ at equal angles. Moreover, the segment $[\zeta_1, \zeta_2] \subset \mathcal{N}$, which is straightforward to verify by definition of $\mathcal{N}$.

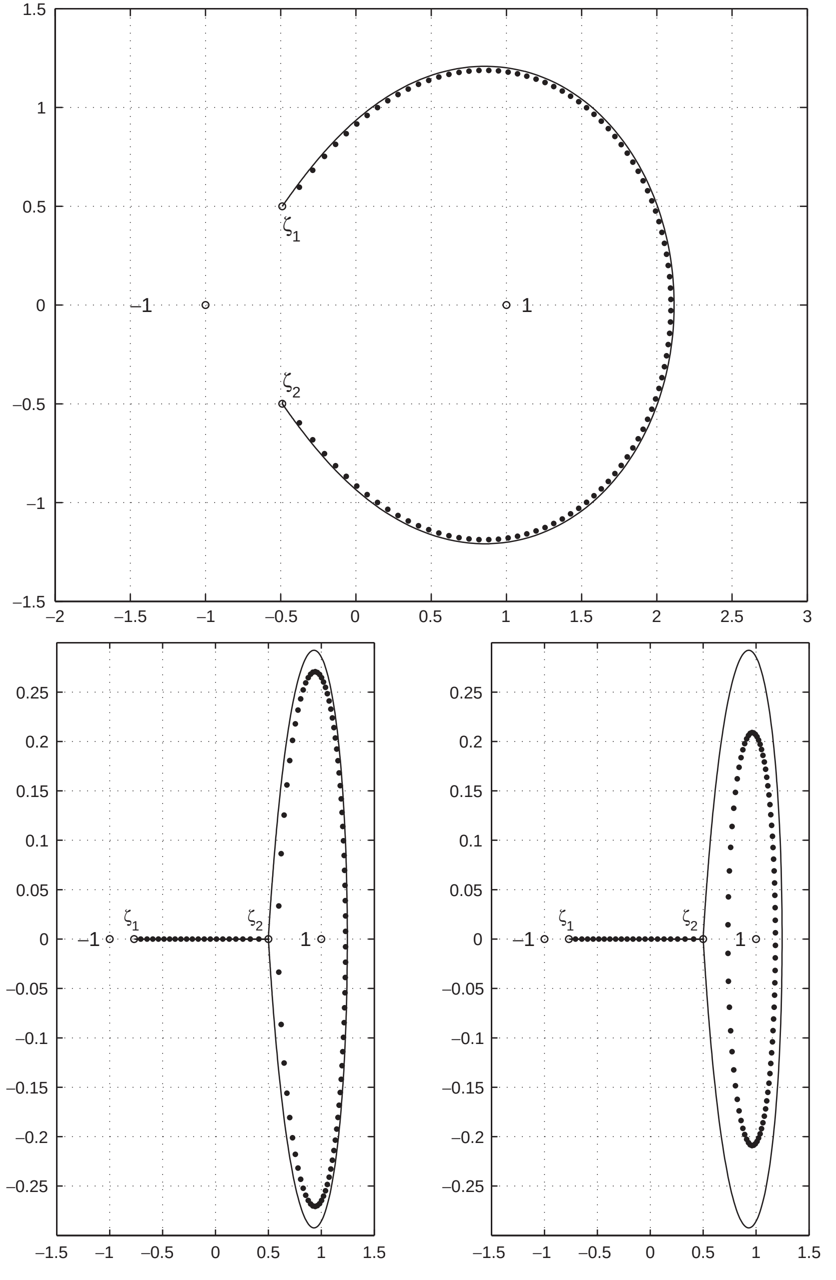

Fig. 5. Up, case C.2: zeros of $P_{100}^{(-110-10^{-5},50-10^{-5})}$, along with the curve Γ, corresponding to $A=-1.1$ and $B=0.5$. Down, case C.3: zeros of $P_{100}^{(-80-10^{-5},50-10^{-5})}$ (left) and $P_{100}^{(-80-10^{-15},50-10^{-5})}$ (right), along with the set Σ, corresponding to $A=-0.8$ and $B=0.5$.

On the other hand, (2.3) has double poles at $z = \pm 1$ and $z = \infty$, in such a way that if we consider the rational function $Q(z) = -(z - \zeta_1)(z - \zeta_2)(z^2 - 1)^{-2}$, we have that the residues of $\sqrt{Q}$ at these points are purely imaginary. Therefore, we conclude that near these double poles the trajectories are simple closed contours.

Now, the symmetry of Q with respect to the real axis, along with the facts that the trajectories cannot tend to infinity and (2.3) has no other singular point, allows one to ensure that the other critical trajectories are two closed contours emanating from ζ_1 and ζ_2. The fact that a closed trajectory needs to surround a singular point implies that these closed trajectories intersect the real axis in two points, one of them in $(1, +\infty)$ and the other in $(-\infty, -1)$. $\square$

Proof of Lemma 2. Taking into account the definition of $R(z)$ and (2.1), it is easy to see that

$$R(1) = \begin{cases} \dfrac{2A}{A+B+2} < 0 & \text{if } A < -1 < A+B \text{ (case C.2)}, \\ -\dfrac{2A}{A+B+2} > 0 & \text{if } -1 < A < 0 < B \text{ (case C.3)}. \end{cases} \tag{3.1}$$

In the same way,

$$R(-1) = -\frac{2B}{A+B+2} < 0 \quad \text{if } A < 0 < B \quad \text{and} \quad A + B > -1. \tag{3.2}$$

Thus, the definition of Σ in (2.4) yields that (2.5) is real-valued on Σ and does not change sign on each of its components. The same remains valid when $-1 < A < 0 < B$ and $0 < r \leqslant +\infty$ for μ_r on its support $\Sigma_r = \Gamma_r \cup [\zeta_1, \zeta_2]$.

Moreover, for (A, B) such that $-1 < A < 0 < B$, taking into account the definition of Σ and Σ_r, the residue theorem and (3.1)–(3.2), we have for $0 \leqslant r < +\infty$:

$$\begin{aligned} \mu_r(\Gamma_r) &= \int_{\Gamma_r} d\mu_r(t) = -(A+B+2) \operatorname*{res}_{z=1} \left(\frac{R(z)}{z^2-1} \right) \\ &= (A+B+2) \frac{R(1)}{2} = -A; \end{aligned}$$

clearly, also $\mu_\infty(\Gamma_\infty) = \mu_\infty(\{1\}) = -A$.

On the other hand, for $0 \leqslant r \leqslant +\infty$,

$$\begin{aligned} \mu_r([\zeta_1, \zeta_2]) &= \int_{\zeta_1}^{\zeta_2} d\mu_r(t) \\ &= \frac{A+B+2}{2} \left[\operatorname*{res}_{z=1} \left(\frac{R(z)}{z^2-1} \right) + \operatorname*{res}_{z=-1} \left(\frac{R(z)}{z^2-1} \right) + \operatorname*{res}_{z=\infty} \left(\frac{R(z)}{z^2-1} \right) \right] \\ &= \frac{A+B+2}{2} \left(1 + \frac{A}{A+B+2} - \frac{B}{A+B+2} \right) = 1 + A, \end{aligned}$$

and, therefore,

$$\mu_r(\Sigma_r) = \int_{\Sigma_r} d\mu_r(t) = \int_{\Gamma_r} d\mu_r(t) + \int_{\zeta_1}^{\zeta_2} d\mu_r(t) = 1.$$

Analogously, for (A, B) such that $A < -1 < A + B$, it is easy to see that

$$\mu(\Gamma) = \int_{\Gamma} d\mu(t) = 1,$$

and it settles the proof. □

4. Riemann–Hilbert analysis

4.1. Orthogonality and the Riemann–Hilbert problem

As it was mentioned in the Introduction, the key fact for the asymptotic analysis is a full system of orthogonality relations satisfied by the Jacobi polynomials on simple contours, which allows to pose a matrix Riemann–Hilbert problem (RHP) and apply the Deift–Zhou steepest descent method.

The following result was established in [25, Theorem 5.1]:

Lemma 3. *Let $\mathcal{C}$ be a Jordan arc connecting $z = -1+0i$ with $z = -1-0i$ and surrounding $z = 1$ once. If $\beta > 0$, then we have*

$$\int_{\mathcal{C}} t^k P_n^{(\alpha,\beta)}(t) w^2(t; \alpha, \beta)\, dt \begin{cases} = 0, & k < n, \\ \neq 0, & k = n, \end{cases} \tag{4.1}$$

where $w(\cdot; \alpha, \beta)$ has been introduced in (2.11).

From the seminal work of Fokas et al. [19] (see also [7]) it is known that the orthogonality (4.1) can be characterized in terms of the following Riemann–Hilbert problem: find a matrix valued function $Y : \mathbb{C}\setminus\mathcal{C} \to \mathbb{C}^{2\times 2}$ satisfying the conditions below:

(RH1.1) Y is analytic in $\mathbb{C} \setminus \mathcal{C}$.
(RH1.2) Y has continuous boundary values on $\mathcal{C}$, denoted by Y_+ and Y_-, such that

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w^2(z; \alpha, \beta) \\ 0 & 1 \end{pmatrix}, \quad \text{for } z \in \mathcal{C}.$$

(RH1.3) As $z \to \infty$,

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

(RH1.4) Y is bounded in a neighborhood of $z = -1$.

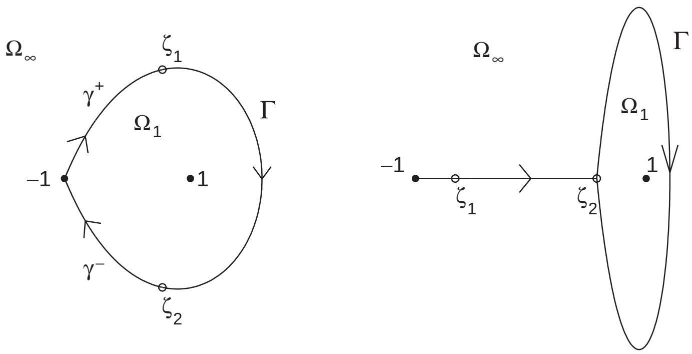

Fig. 6. Contours $\mathcal{C}$ for the Riemann–Hilbert problem for Y; cases C.2 (left) and C.3.

Proposition 1 (*Kuijlaars et al. [25]*). *The unique solution of the Riemann–Hilbert problem* (RH1.1)–(RH1.4) *is given by*

$$Y(z) = \begin{pmatrix} p_n(z) & \dfrac{1}{2\pi i}\displaystyle\int_{\mathcal{C}} \dfrac{p_n(t)w^2(t;\alpha,\beta)}{t-z}\,dt \\ q_{n-1}(z) & \dfrac{1}{2\pi i}\displaystyle\int_{\mathcal{C}} \dfrac{q_{n-1}(t)w^2(t;\alpha,\beta)}{t-z}\,dt \end{pmatrix},$$

where $p_n(z) = \widehat{P}_n^{(\alpha,\beta)}(z)$ *is the monic Jacobi polynomial, and* $q_{n-1}(z) = b_{n-1}P_{n-1}^{(\alpha,\beta)}(z)$, *for some suitable non-zero constant* b_{n-1}.

Let (A, B) be a pair satisfying (1.7). Then for every $n \in \mathbb{N}$, monic polynomials $\widehat{P}_n^{(An,Bn)}$ satisfy the conditions of Lemma 3. Hence, for any $\mathcal{C}$ as described in Lemma 3, polynomials $\widehat{P}_n^{(An,Bn)}$ can be characterized as the (1, 1) entry of the unique matrix Y solving the Riemann–Hilbert problem (RH1.1)–(RH1.4) with

$$\alpha = An, \quad \beta = Bn.$$

Taking advantage of the freedom in the selection of the contour $\mathcal{C}$ in (4.1) we will choose different configurations for both cases C.2 and C.3. This choice is mainly suggested by the numerical evidence on the actual location of zeros. For describing the appropriate $\mathcal{C}$ we will use the contours defined in subsection 2.1, corresponding to A and B fixed.

In case C.2, when $A < -1 < A + B$, we will make the contour $\mathcal{C}$ in Lemma 3 coincide with $\gamma^- \cup \Gamma \cup \gamma^+$, oriented clockwise (Fig. 6, left). Hence, we are interested in the Riemann–Hilbert problem (RH1.1)–(RH1.4) with $\mathcal{C} = \gamma^- \cup \Gamma \cup \gamma^+$, $\alpha = An$ and $\beta = Bn$.

However, in case C.3, when $-1 < A < 0 < B$, it is convenient to make part of the original contour $\mathcal{C}$ coalesce along the interval $[-1, \zeta_2]$ (traversed twice in opposite directions), and the rest go along the arc Γ. This deformation creates a new contour, which we denote again by $\mathcal{C}$, and we choose its orientation as in Fig. 6, right. Consequently, it yields a new RHP, still characterizing the polynomials p_n. With respect to problem (RH1.1)–(RH1.4), we have to modify only the jump matrix on $(-1, \zeta_2)$ as a result of the coalescence of two original sub-arcs of $\mathcal{C}$: its (1, 2) entry becomes $w_+^{2n} - w_-^{2n}$ on $(-1, \zeta_2)$. The new Riemann–Hilbert problem is: find a matrix valued function $Y \equiv Y^{(A,B)} : \mathbb{C} \setminus ([-1, \zeta_2] \cup \Gamma) \to \mathbb{C}^{2\times 2}$ such that the following conditions hold:

(RH2.1) Y is analytic on $\mathbb{C} \setminus ([-1, \zeta_2] \cup \Gamma)$.

(RH2.2) Y has continuous boundary values on $\mathcal{C} \setminus \{-1, \zeta_2\}$, denoted by Y_+ and Y_-, such that $Y_+(z) = Y_-(z) J_Y(z)$, where

$$J_Y(z) = \begin{cases} \begin{pmatrix} 1 & w^{2n}(z) \\ 0 & 1 \end{pmatrix}, & z \in \Gamma, \\ \begin{pmatrix} 1 & d_n\, w_+^{2n}(z) \\ 0 & 1 \end{pmatrix}, & z \in (-1, \zeta_2), \end{cases}$$

with

$$d_n \stackrel{\text{def}}{=} 1 - e^{-2An\pi i} = 2ie^{-An\pi i} \, \sin(An\pi), \tag{4.2}$$

and with $w(z) = w(z; A, B)$ defined in (2.11).

(RH2.3) As $z \to \infty$,

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

(RH2.4) Y is bounded in a neighborhood of $z = -1$ and $z = \zeta_2$.

In both cases $\mathbb{C} \setminus \mathcal{C}$ has two connected components, one containing $z = 1$ and the other containing infinity; we denote these components by Ω_1 and Ω_∞, respectively (see Fig. 6).

The steepest descent analysis, that we are going to carry out next, introduces new contours which are unions of a finite number of curves and arcs on $\mathbb{C}$. In order to simplify notation we will call all the end points and points of self-intersection of such curves *singular* points, and the rest will be *regular* points of the contour. Hence, we could rephrase (RH1.4) and (RH2.4) saying that Y is bounded in a neighborhood of all singular points of $\mathcal{C}$.

4.2. *First transformation $Y \mapsto U$*

In order to shorten notation, we use the Pauli matrix

$$\sigma_3 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and denote $x^{\sigma_3} = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$. Also for the sake of brevity, it is convenient to introduce the function

$$H(z) \stackrel{\text{def}}{=} G(z) w(z), \tag{4.3}$$

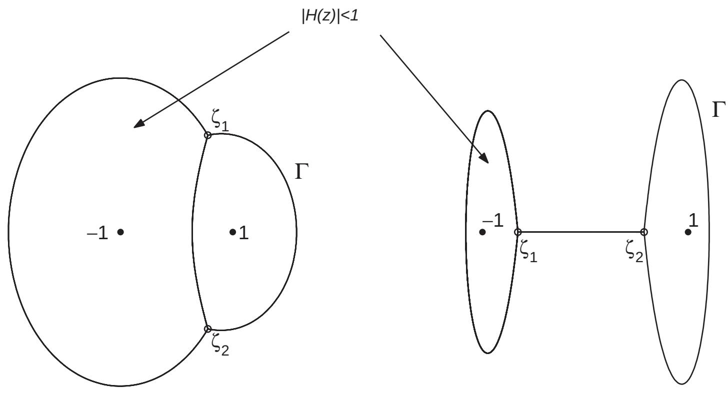

Fig. 7. Regions where $|H(z)| < 1$ for cases C.2 (left) and C.3.

analytic and single-valued in $\mathbb{C} \setminus (\Sigma \cup (-\infty, 1])$. Using (2.6)–(2.12), we immediately get that

- For case C.2,

$$H(z) = \exp\left(\frac{A+B+2}{2}\int_{\zeta_2}^{z}\frac{R(t)}{t^2-1}dt\right), \quad \text{for } z \in \mathbb{C} \setminus (\Gamma \cup (-\infty, 1])\,. \tag{4.4}$$

- For case C.3,

$$H(z) = \begin{cases} \exp\left(\dfrac{A+B+2}{2}\displaystyle\int_{\zeta_2+i0}^{z}\frac{R(t)}{t^2-1}dt\right), & \text{for } z \in \mathbb{C} \setminus (\mathrm{Pc}(\Sigma) \\ & \quad \cup(-\infty, 1])\,; \\ \exp\left(-\dfrac{A+B+2}{2}\displaystyle\int_{\zeta_2}^{z}\frac{R(t)}{t^2-1}dt\right), & \text{for } z \in \mathrm{Int}(\mathrm{Pc}(\Sigma)) \cap \mathbb{C}^+, \\ e^{-\pi i A}\exp\left(-\dfrac{A+B+2}{2}\displaystyle\int_{\zeta_2}^{z}\frac{R(t)}{t^2-1}dt\right), & \text{for } z \in \mathrm{Int}(\mathrm{Pc}(\Sigma)) \cap \mathbb{C}^-. \end{cases} \tag{4.5}$$

Observe that the same convention as in (2.9) for the path of integration applies:

$$\lim_{\mathbb{C}^+\setminus\Sigma\ni z\to\zeta_2} H(z) = 1.$$

Furthermore, taking into account (2.26), in the case C.3,

$$\lim_{\mathbb{C}^-\setminus\mathrm{Pc}(\Sigma)\ni z\to\zeta_2} H(z) = e^{-\pi i\mu(\Gamma)} = e^{A\pi i}. \tag{4.6}$$

In both cases C.2 and C.3, the sets of trajectories $\mathcal{N}$ and $\mathcal{N}_r$, $(r \geqslant 0)$, introduced in Section 2.3, may be characterized by the conditions $|H(z)| = 1$ and $|H(z)| = e^{r/2}$, respectively. Fig. 7 shows also the domains where $|H(z)| < 1$.

Function H has continuous boundary values at regular points of $\Sigma \cup (-\infty, 1]$, which satisfy:

- In case C.2:

$$H_+(z) = \begin{cases} H_-^{-1}(z), & z \in \Gamma, \\ e^{\pi i A}\, H_-(z), & z \in (-1, 1), \\ e^{\pi i (A+B)}\, H_-(z), & z \in (-\infty, -1). \end{cases}$$

- In case C.3:

$$H_+(z) = \begin{cases} H_-^{-1}(z), & z \in \Gamma, \\ e^{-\pi i A} H_-^{-1}(z), & z \in (\zeta_1, \zeta_2), \\ e^{\pi i A}\, H_-(z), & z \in (-1, 1) \setminus [\zeta_1, \zeta_2], \\ e^{\pi i (A+B)}\, H_-(z), & z \in (-\infty, -1). \end{cases} \tag{4.7}$$

This allows us to express the boundary values of G at the regular points of Σ in terms of H:

$$\frac{G_+(z)}{G_-(z)} = H_+^2(z) \quad \text{and} \quad G_+(z)G_-(z) = \frac{1}{w_+^2(z)}. \tag{4.8}$$

Now we are ready to introduce the first transformation of the RHP, with the aim to normalize it at infinity. For d_n in (4.2), let us fix any value of $d_n^{1/2}$, and define

$$U(z) = \begin{cases} \kappa^{n\sigma_3} Y(z) G(z)^{-n\sigma_3}, & \text{in case C.2;} \\ d_n^{-\sigma_3/2} \kappa^{n\sigma_3} Y(z) G(z)^{-n\sigma_3} d_n^{\sigma_3/2}, & \text{in case C.3,} \end{cases} \tag{4.9}$$

with κ given by (2.10). Obviously, matrix U solves now a new Riemann–Hilbert problem. Taking into account (4.8) we can state it as:

(RH3.1) U is analytic on $\mathbb{C} \setminus \mathcal{C}$.

(RH3.2) U has continuous boundary values at the regular points of $\mathcal{C}$, denoted by U_+ and U_-, such that $U_+(z) = U_-(z) J_U(z)$, where, for (A, B) in case C.2,

$$J_U = \begin{cases} \begin{pmatrix} H_+^{-2n}(z) & 1 \\ 0 & H_+^{2n}(z) \end{pmatrix}, & z \in \Gamma, \\ \begin{pmatrix} 1 & H^{2n}(z) \\ 0 & 1 \end{pmatrix}, & z \in \gamma^+ \cup \gamma^-, \end{cases} \tag{4.10}$$

and for (A, B) in case C.3,

$$J_U(z) = \begin{cases} \begin{pmatrix} H_+^{-2n}(z) & d_n^{-1} \\ 0 & H_+^{2n}(z) \end{pmatrix}, & z \in \Gamma; \\ \begin{pmatrix} H_+^{-2n}(z) & 1 \\ 0 & H_+^{2n}(z) \end{pmatrix}, & z \in (\zeta_1, \zeta_2); \\ \begin{pmatrix} 1 & H_+^{2n}(z) \\ 0 & 1 \end{pmatrix}, & z \in (-1, \zeta_1). \end{cases} \tag{4.11}$$

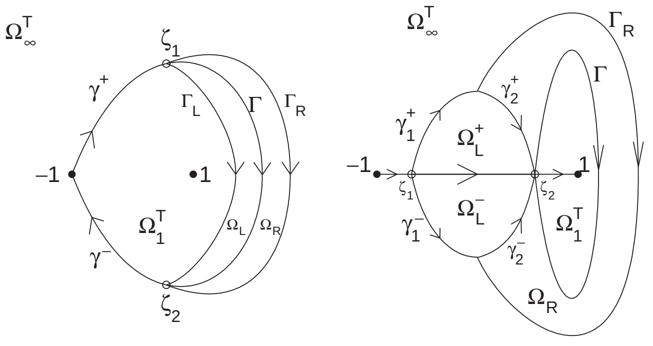

Fig. 8. Contours $\mathcal{C}^T$ for T.

(RH3.3) As $z \to \infty$,

$$U(z) = I + O\left(\frac{1}{z}\right).$$

(RH3.4) Matrix U is bounded in a neighborhood of the singular points of $\mathcal{C}$.

4.3. Second transformation $U \mapsto T$

By (4.10) and (4.11), the jump matrix J_U has oscillatory diagonal entries on Σ, along with exponentially decaying (as $n \to \infty$) off-diagonal entries elsewhere and away from $\zeta_{1,2}$ (see Fig. 7). The aim of the next step is to transform the jump matrices with oscillatory diagonal entries into matrices asymptotically close to the identity matrix or to matrices with constant jumps. To this end, we take advantage of an appropriate factorization of J_U and "open the lenses" around contours $\mathcal{C}$.

In case C.2, we use the following factorization of the jump matrix for $z \in \Gamma$ (where we have taken into account that $H_+ = 1/H_-$ on Γ):

$$J_U(z) = \begin{pmatrix} 1 & 0 \\ H_-^{-2n}(z) & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ H_+^{-2n}(z) & 1 \end{pmatrix}. \tag{4.12}$$

Thus, the problem of the oscillatory diagonal entries of the jump matrix for $z \in \Gamma$ may be solved by opening the lenses around Γ as it is shown in Fig. 8 (left). The new contours Γ_L and Γ_R are also oriented from ζ_1 to ζ_2, and this gives us two new bounded regions, Ω_R and Ω_L, as well as modified domains $\Omega_1^T \stackrel{\text{def}}{=} \Omega_1 \setminus \overline{\Omega_L}$ and $\Omega_\infty^T \stackrel{\text{def}}{=} \Omega_\infty \setminus \overline{\Omega_R}$; we also denote $\mathcal{C}^T \stackrel{\text{def}}{=} \mathcal{C} \cup \Gamma_L \cup \Gamma_R$, with the orientation shown in Fig. 8, left.

Hence, taking into account (4.12), we define in case C.2 a new matrix-valued function $T : \mathbb{C} \setminus \mathcal{C}^T \longrightarrow \mathbb{C}^{2\times 2}$ by

$$T(z) = U(z) \cdot \begin{cases} I, & \text{for } z \in \Omega_1^T \cup \Omega_\infty^T, \\ \begin{pmatrix} 1 & 0 \\ H^{-2n}(z) & 1 \end{pmatrix}, & \text{for } z \in \Omega_L, \\ \begin{pmatrix} 1 & 0 \\ -H^{-2n}(z) & 1 \end{pmatrix}, & \text{for } z \in \Omega_R. \end{cases} \tag{4.13}$$

It solves the following RHP:

(RH4.1) T is analytic for $z \in \mathbb{C} \setminus \mathcal{C}^T$;
(RH4.2) $T(z)$ possesses continuous boundary values at regular points of $\mathcal{C}^T$, T_+ and T_-, related by the following jump conditions:

$$T_+(z) = T_-(z) J_T(z), \quad z \in \mathcal{C}^T,$$

where the jump matrix J_T is

$$J_T(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \Gamma, \\ \begin{pmatrix} 1 & H^{2n}(z) \\ 0 & 1 \end{pmatrix}, & z \in \gamma^+ \cup \gamma^-, \\ \begin{pmatrix} 1 & 0 \\ H^{-2n}(z) & 1 \end{pmatrix}, & z \in \Gamma_L \cup \Gamma_R. \end{cases}$$

(RH4.3) $T(z)$ has the following behavior at infinity:

$$T(z) = I + O(1/z) \quad \text{as } z \to \infty.$$

(RH4.4) $T(z)$ is bounded in a neighborhood of the singular points of $\mathcal{C}^T$.

In principle, we could take advantage of factorization (4.12) also in the case C.3. However, the geometry here is more complicated; this procedure would eventually yield a constant jump on whole Σ, which has now two components. In order to give a unified treatment to both cases in the next step, we use now a different factorization for J_U:

$$J_U(z) = \begin{cases} \begin{pmatrix} 0 & d_n^{-1} \\ -d_n & H_-^{-2n}(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d_n H_+^{-2n}(z) & 1 \end{pmatrix}, & z \in \Gamma, \\ \begin{pmatrix} 1 & 0 \\ e^{-2A\pi n i} H_-^{-2n}(z) & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ H_+^{-2n}(z) & 1 \end{pmatrix}, & z \in (\zeta_1, \zeta_2). \end{cases}$$

These factorizations suggest to open lenses in the way shown in Fig. 8, right, which yields the new contour $\mathcal{C}^T$, splitting $\mathbb{C}$ into domains Ω_1^T, Ω_∞^T, $\Omega_L^\pm$, and Ω_R, as shown.

Now we define the matrix-valued function $T : \mathbb{C} \setminus \mathcal{C}^T \to \mathbb{C}^{2\times 2}$ in the following way:

$$T(z) = U(z) \cdot \begin{cases} \begin{pmatrix} 1 & 0 \\ -d_n H^{-2n}(z) & 1 \end{pmatrix}, & z \in \Omega_R, \\ \begin{pmatrix} 0 & d_n^{-1} \\ -d_n & H^{-2n}(z) \end{pmatrix}, & z \in \Omega_1^T, \\ \begin{pmatrix} 1 & 0 \\ -H^{-2n}(z) & 1 \end{pmatrix}, & z \in \Omega_L^+, \\ \begin{pmatrix} 1 & 0 \\ e^{-2A\pi n i} H^{-2n}(z) & 1 \end{pmatrix}, & z \in \Omega_L^-, \\ I, & z \in \Omega_\infty^T. \end{cases} \tag{4.14}$$

Then $T(z)$ solves problem (RH4.1)–(RH4.4), but with the expression of J_T replaced by

$$J_T(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (\zeta_1, \zeta_2), \\ \begin{pmatrix} 1 & 0 \\ d_n H^{-2n}(z) & 1 \end{pmatrix}, & z \in \Gamma_R, \\ \begin{pmatrix} 1 & 0 \\ H^{-2n}(z) & 1 \end{pmatrix}, & z \in \gamma_1^+ \cup \gamma_2^-, \\ \begin{pmatrix} 1 & 0 \\ e^{-2A\pi n i} H^{-2n}(z) & 1 \end{pmatrix}, & z \in \gamma_1^- \cup \gamma_2^+, \\ \begin{pmatrix} 1 & H_+^{2n}(z) \\ 0 & 1 \end{pmatrix}, & z \in (-1, \zeta_1), \\ \begin{pmatrix} 1 & e^{2A\pi n i} H_+^{-2n}(z) \\ 0 & 1 \end{pmatrix}, & z \in (\zeta_2, 1), \\ I, & z \in \Gamma. \end{cases}$$

4.4. Construction of the parametrices

Now, we can see (cf. Fig. 7) that we have a single open arc joining the branch points (Γ in case C.2, and (ζ_1, ζ_2) in case C.3) where the jump matrix J_T is constant, and at a positive distance from these arcs, J_T is asymptotically exponentially close to the identity matrix. Hence, by ignoring the "close-to-identity" jumps and condition (RH4.4) we are lead to the following problem: find an analytic matrix-valued function $N(z) = I + O(1/z)$, $z \to \infty$, and having the jump

$$N_+(z) = N_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

on Γ (in case C.2) or on (ζ_1, ζ_2) in case C.3, with the orientation "from ζ_1 to ζ_2" chosen. A solution of this model RHP, which is not unique in general, is (cf. [7, Chapter 7]):

$$N(z) = \begin{pmatrix} \dfrac{a(z) + a(z)^{-1}}{2} & \dfrac{a(z) - a(z)^{-1}}{2i} \\ -\dfrac{a(z) - a(z)^{-1}}{2i} & \dfrac{a(z) + a(z)^{-1}}{2} \end{pmatrix}, \tag{4.15}$$

where a has been defined in (2.14); it satisfies

$$N(z) = O(|z - \zeta_j|^{-1/4}), \quad z \to \zeta_j, \quad j = 1, 2,$$

showing that the singularities at ζ_j are L^2-integrable. Observe that the $(1, 1)$ and $(1, 2)$ entries of N coincide with N_{11} and N_{12}, introduced in (2.13).

We may expect N to be close to T away from ζ_1 and ζ_2. However, in a neighborhood of the branch points the ignored jumps are no longer close to identity, and a different parametrix (model problem) is required. Now we look for two matrices $P^{(j)}$, $j \in \{1, 2\}$, which have the same jumps as T in a neighborhood of $z = \zeta_j$, and match N on the boundary of these neighborhoods.

The construction of these matrices is well described for instance in [7]. Denote by $\Delta_\varepsilon(s) \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z - s| < \varepsilon\}$, where we take $\varepsilon > 0$ sufficiently small. A local parametrix $P^{(j)}$ in $\Delta_\varepsilon(\zeta_j)$, $j \in \{1, 2\}$, solves the RHP with the same jumps as T there (see Fig. 9):

(RH5.1) $P^{(j)}$ is analytic for $z \in \Delta_\varepsilon(\zeta_j) \setminus \mathcal{C}^T$, bounded and continuous in $\overline{\Delta_\varepsilon(\zeta_j)} \setminus \mathcal{C}^T$;

(RH5.2) $P^{(j)}(z)$ possesses continuous boundary values at regular points of $\mathcal{C}^T \cap \Delta_\varepsilon(\zeta_j)$, $P_+^{(j)}$ and $P_-^{(j)}$, related by the following jump conditions:

$$P_+^{(j)}(z) = P_-^{(j)}(z) J_{P^{(j)}}(z), \quad z \in \mathcal{C}^T \cap \Delta_\varepsilon(\zeta_j).$$

(RH5.3) there exists a constant $M > 0$ such that for every $z \in \partial\Delta_\varepsilon(\zeta_j) \setminus \mathcal{C}^T$,

$$\| P^{(j)}(z) N^{-1}(z) - I \| \leqslant \frac{M}{n}.$$

We describe the construction for ζ_2; in order to simplify notation we write P instead of $P^{(2)}$ whenever it cannot lead us into confusion. The jumps $J_P = J_{P^{(2)}}$ specified in (RH5.2) are

- In case C.2:

$$J_P(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \Gamma \cap \Delta_\varepsilon(\zeta_2), \\ \begin{pmatrix} 1 & 0 \\ H^{-2n}(z) & 1 \end{pmatrix}, & z \in (\Gamma_L \cup \Gamma_R) \cap \Delta_\varepsilon(\zeta_2), \\ \begin{pmatrix} 1 & H^{2n}(z) \\ 0 & 1 \end{pmatrix}, & z \in \gamma^- \cap \Delta_\varepsilon(\zeta_2). \end{cases}$$

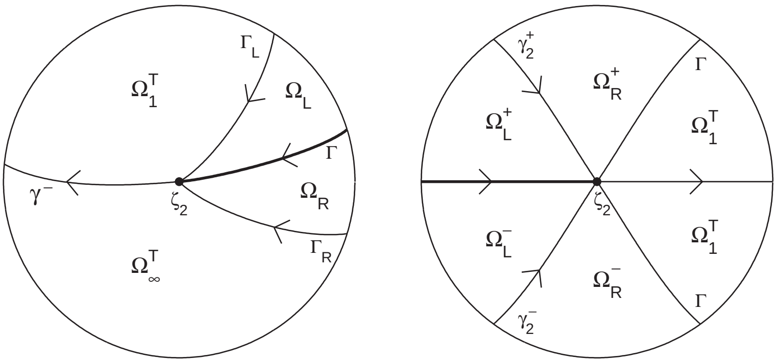

Fig. 9. Local analysis for cases C.2 (left) and C.3.

- In case C.3:

$$J_P(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (\zeta_2 - \varepsilon, \zeta_2), \\ \begin{pmatrix} 1 & 0 \\ e^{-2A\pi n i} H^{-2n}(z) & 1 \end{pmatrix}, & z \in \gamma_2^+ \cap \Delta_\varepsilon(\zeta_2), \\ \begin{pmatrix} 1 & 0 \\ H^{-2n}(z) & 1 \end{pmatrix}, & z \in \gamma_2^- \cap \Delta_\varepsilon(\zeta_2), \\ \begin{pmatrix} 1 & e^{2A\pi n i} H_+^{-2n}(z) \\ 0 & 1 \end{pmatrix}, & z \in (\zeta_2, \zeta_2 + \varepsilon), \\ I, & z \in \Gamma. \end{cases}$$

In order to solve the Riemann–Hilbert problems for P, let us first make a simple change of functions yielding piecewise constant jump matrices. For this purpose, we set for $z \in \overline{\Delta_\varepsilon(\zeta_2)} \setminus \mathcal{C}^T$,

$$R(z) \stackrel{\text{def}}{=} P(z) \cdot \begin{cases} e^{-n\phi(z)\sigma_3}, & \text{in case C.2}, \\ e^{A\pi i n \sigma_3}\, e^{-n\phi(z)\sigma_3}, & \text{in case C.3}, \end{cases} \tag{4.16}$$

where ϕ is the function introduced in (2.21). In order to compute the new jumps we need to find how ϕ is related to H. In case C.2, by (4.4), $\exp(-\phi(z)) = H(z)$ for $z \in \Delta_\varepsilon(\zeta_2) \setminus \Gamma$. In case C.3, by continuity of ϕ in $\Delta_\varepsilon(\zeta_2) \setminus (\zeta_1, \zeta_2)$,

$$\exp(-\phi(z)) = \begin{cases} H(z), & z \in (\Omega_L \cup \Omega_R) \cap \Delta_\varepsilon(\zeta_2) \cap \mathbb{C}^+, \\ H^{-1}(z), & z \in \Omega_1^T \cap \Delta_\varepsilon(\zeta_2) \cap \mathbb{C}^+, \\ e^{-\pi i A}\, H^{-1}(z), & z \in \Omega_1^T \cap \Delta_\varepsilon(\zeta_2) \cap \mathbb{C}^-, \\ e^{-\pi i A}\, H(z)\,, & z \in (\Omega_L \cup \Omega_R) \cap \Delta_\varepsilon(\zeta_2) \cap \mathbb{C}^- \end{cases} \tag{4.17}$$

and

$$(\phi_+ + \phi_-)(z) = 2\pi i A, \quad z \in (\zeta_2 - \varepsilon, \zeta_2).$$

Now we can compute the jump matrix for R: $J_R = e^{n\phi_-(z)\sigma_3} J_T e^{-n\phi_+(z)\sigma_3}$, namely:

- In case C.2,

$$J_R(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \Gamma \cap \Delta_\varepsilon(\zeta_2), \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in (\Gamma_L \cup \Gamma_R) \cap \Delta_\varepsilon(\zeta_2), \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & z \in \gamma_- \cap \Delta_\varepsilon(\zeta_2). \end{cases}$$

- In case C.3,

$$J_R(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (\zeta_2 - \varepsilon, \zeta_2), \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in \gamma_2^\pm \cap \Delta_\varepsilon(\zeta_2), \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & z \in (\zeta_2, \zeta_2 + \varepsilon). \end{cases}$$

Observe now that we have essentially the same local problem in both cases; in this way, we have reduced the RHP to the one studied in [8] (see also [7, Chapter 7]), and we can write its solution explicitly.

For $\varepsilon > 0$ small enough, function

$$f(z) = \tfrac{3}{2}\, \left(\phi(z)\right)^{2/3},$$

defined in (2.22), is a conformal mapping from the neighborhood of the branch point onto a neighborhood of 0. In case C.2, Γ and γ^- are mapped onto the negative and positive real axis, respectively, and in case C.3 it happens to (ζ_1, ζ_2) and $(\zeta_2, 1)$. Also we may deform the other curves (Γ_L and Γ_R in case C.2, and $\gamma_2^\pm$ in case C.3) in such a way that the points on their image by f close to the branch point have the argument $\pm 2\pi/3$.

Then the problem for R is solved by

$$R(z) = \Psi\left(n^{2/3} f(z)\right),$$

where Ψ is built out of the Airy function Ai (see e.g. [1]) and its derivative Ai$'$ as follows:

$$\Psi(t) = \begin{cases} \begin{pmatrix} \mathrm{Ai}(t) & \mathrm{Ai}(\omega^2 t) \\ \mathrm{Ai}'(t) & \omega^2 \mathrm{Ai}'(\omega^2 t) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & 0 < \arg t < 2\pi/3; \\ \begin{pmatrix} \mathrm{Ai}(t) & \mathrm{Ai}(\omega^2 t) \\ \mathrm{Ai}'(t) & \omega^2 \mathrm{Ai}'(\omega^2 t) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & 2\pi/3 < \arg t < \pi; \\ \begin{pmatrix} \mathrm{Ai}(t) & -\omega^2 \mathrm{Ai}(\omega t) \\ \mathrm{Ai}'(t) & -\mathrm{Ai}'(\omega t) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & -\pi < \arg t < -2\pi/3; \\ \begin{pmatrix} \mathrm{Ai}(t) & -\omega^2 \mathrm{Ai}(\omega t) \\ \mathrm{Ai}'(t) & -\mathrm{Ai}'(\omega t) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & -2\pi/3 < \arg t < 0, \end{cases} \tag{4.18}$$

and $\omega \stackrel{\text{def}}{=} e^{2\pi i/3}$. Finally, matrix P solving (RH5.1)–(RH5.3) is

$$P(z) = E(z)\, R(z) \cdot \begin{cases} e^{n\phi(z)\sigma_3}, & \text{in case C.2,} \\ e^{-A\pi i n\sigma_3}\, e^{n\phi(z)\sigma_3}, & \text{in case C.3,} \end{cases} \tag{4.19}$$

where the analytic matrix function E is

$$E(z) \stackrel{\text{def}}{=} \sqrt{\pi} e^{\frac{\pi i}{6}} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \left(\frac{n^{1/6} f(z)^{1/4}}{a(z)} \right)^{\sigma_3}. \tag{4.20}$$

4.5. *Final transformation $T \mapsto S$*

Now we may use matrix valued functions N and $P^{(j)}$ for the final transformation. Recalling the definition of the contour $\mathcal{C}^T$, define the matrix-valued function S:

$$S(z) \stackrel{\text{def}}{=} \begin{cases} T(z)N(z)^{-1}, & z \in \mathbb{C} \setminus (\mathcal{C}^T \cup \overline{\Delta_\varepsilon(\zeta_1)} \cup \overline{\Delta_\varepsilon(\zeta_2)}), \\ T(z)\left(P^{(j)}(z)\right)^{-1}, & z \in \Delta_\varepsilon(\zeta_j), \quad j = 1, 2. \end{cases} \tag{4.21}$$

It is immediate to check that S is analytic in $\mathbb{C} \setminus \mathcal{C}^S$, where $\mathcal{C}^S$ is the contour shown in Fig. 10. Moreover, $S : \mathbb{C} \setminus \mathcal{C}^S \longrightarrow \mathbb{C}^{2\times 2}$ satisfies the following Riemann–Hilbert problem:

(RH6.1) S is analytic in $\mathbb{C} \setminus \mathcal{C}^S$.
(RH6.2) S has continuous boundary values at regular points of $\mathcal{C}^S$, denoted by S_+ and S_-, such that $S_+(z) = S_-(z) J_S(z)$, where

$$J_S(z) = \begin{cases} P^{(j)}(z)N(z)^{-1}, & z \in \partial\Delta_\varepsilon(\zeta_j), \quad j = 1, 2, \\ N(z)J_T(z)N(z)^{-1}, & z \in \mathcal{C}^S \setminus (\partial\Delta_\varepsilon(\zeta_1) \cup \partial\Delta_\varepsilon(\zeta_2)). \end{cases}$$

(RH6.3) $S(z) = I + O(1/z)\,, \ z \to \infty$.
(RH6.4) $S(z)$ is bounded in a neighborhood of the singular points of $\mathcal{C}^S$.

Observe that by construction, $J_S = I + O(1/n)$ as $n \to \infty$ on $\partial\Delta_\varepsilon(\zeta_j)$, $j = 1, 2$, and is exponentially close to I on the rest of contours of $\mathcal{C}^S$. Using the same arguments as in [7] we conclude that

$$S(z) = I + O\left(\frac{1}{n}\right)$$

uniformly for $z \in \mathbb{C} \setminus \mathcal{C}^S$.

5. Proofs of the main results

We establish strong asymptotics for $\{p_n\}$ tracing back all the previous transformations. For the sake of brevity, we do it explicitly only for case C.3. The proofs in case C.2 are very similar.

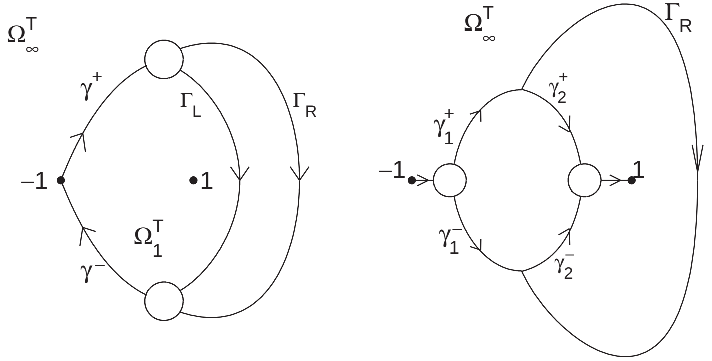

Fig. 10. Contours $\mathcal{C}^S$ for S in cases C.2 (left) and C.3.

5.1. *Proof of Theorems 1 and 2 in case C.3*

Assume that $-1 < A < 0 < B$ (case C.3); by (4.9),

$$Y(z) = d_n^{\sigma_3/2}\kappa^{-n\sigma_3}U(z)G(z)^{n\sigma_3}d_n^{-\sigma_3/2},$$

so that

$$p_n(z) = Y_{11}(z) = \left(\frac{G(z)}{\kappa}\right)^n U_{11}(z). \tag{5.1}$$

Assume that $z \in \mathbb{C} \setminus \mathrm{Pc}(\Sigma)$, away from the branch points; without loss of generality we may take $z \in \Omega^T_\infty \setminus (\overline{\Delta_\varepsilon(\zeta_1)} \cup \overline{\Delta_\varepsilon(\zeta_2)})$ (see Fig. 8 or 10). Then by (4.14),

$$U(z) = S(z)N(z),$$

and taking into account the expression of N in (4.15), we obtain that uniformly on compact subsets of $\Omega^T_\infty \setminus (\overline{\Delta_\varepsilon(\zeta_1)} \cup \overline{\Delta_\varepsilon(\zeta_2)})$,

$$\begin{aligned} p_n(z) = Y_{11}(z) &= \left(\frac{G(z)}{\kappa}\right)^n (SN)_{11}(z) \\ &= \left(\frac{G(z)}{\kappa}\right)^n N_{11}(z)\left(1 + O\left(\frac{1}{n}\right)\right), \end{aligned} \tag{5.2}$$

which proves (2.15).

If z belongs to the bounded component of $\mathbb{C}\setminus\Gamma$, we may assume without loss of generality that $z \in \Omega^T_1 \setminus \overline{\Delta_\varepsilon(\zeta_2)}$. By (4.14) and (4.21),

$$U(z) = S(z)N(z)\begin{pmatrix} H^{-2n}(z) & -d_n^{-1} \\ d_n & 0 \end{pmatrix},$$

and again uniformly in compact subsets of $\Omega_1^T \setminus \Delta_\varepsilon(\zeta_2)$,

$$\begin{aligned} p_n(z) &= \left(\frac{G(z)}{\kappa}\right)^n \left([SN]_{11}(z) H^{-2n}(z) + d_n [SN]_{12}(z)\right) \\ &= \left(\frac{G(z)}{\kappa}\right)^n \left(H^{-2n}(z) N_{11}(z) \left(1 + O\left(\frac{1}{n}\right)\right) + d_n N_{12}(z) \left(1 + O\left(\frac{1}{n}\right)\right)\right), \end{aligned}$$

which proves (2.16). Obviously, this formula is valid also if $z \in \Gamma_-$, that is, if z lies on the "−"-side of Γ, away from the branch points.

Assume now that $z \in \Gamma_+$ away from ζ_2. Again, without loss of generality we may take $z \in \Omega_R \setminus \overline{\Delta_\varepsilon(\zeta_2)}$. By (4.14) and (4.21),

$$U(z) = S(z)N(z) \begin{pmatrix} 1 & 0 \\ d_n H^{-2n}(z) & 1 \end{pmatrix},$$

and uniformly in compact subsets of $\Omega_R \setminus \overline{\Delta_\varepsilon(\zeta_2)}$,

$$\begin{aligned} p_n(z) &= \left(\frac{G(z)}{\kappa}\right)^n \left([SN]_{11}(z) + d_n H^{-2n}(z) [SN]_{12}(z)\right) \\ &= \left(\frac{G(z)}{\kappa}\right)^n \left(N_{11}(z) \left(1 + O\left(\frac{1}{n}\right)\right) + d_n H^{-2n}(z) N_{12}(z) \left(1 + O\left(\frac{1}{n}\right)\right)\right), \end{aligned}$$

which proves (2.18). Using (4.7) it is easy to see that formulas (2.16) and (2.18) match on Γ.

Finally, if z lies on the $\pm$-side of the interval (ζ_1, ζ_2), we assume $z \in \Omega_L^\pm \setminus (\overline{\Delta_\varepsilon(\zeta_1)} \cup \overline{\Delta_\varepsilon(\zeta_2)})$, where by (4.14) and (4.21),

$$U(z) = S(z)N(z) \cdot \begin{cases} \begin{pmatrix} 1 & 0 \\ H^{-2n}(z) & 1 \end{pmatrix}, & z \in \Omega_L^+, \\ \begin{pmatrix} 1 & 0 \\ -e^{-2A\pi i n} H^{-2n}(z) & 1 \end{pmatrix}, & z \in \Omega_L^-. \end{cases}$$

Hence, uniformly in compact subsets of $\Omega_L^+ \setminus (\overline{\Delta_\varepsilon(\zeta_1)} \cup \overline{\Delta_\varepsilon(\zeta_2)})$,

$$\begin{aligned} p_n(z) &= \left(\frac{G(z)}{\kappa}\right)^n \left([SN]_{11}(z) + H^{-2n}(z) [SN]_{12}(z)\right) \\ &= \left(\frac{G(z)}{\kappa}\right)^n \left(N_{11}(z) \left(1 + O\left(\frac{1}{n}\right)\right) + H^{-2n}(z) N_{12}(z) \left(1 + O\left(\frac{1}{n}\right)\right)\right), \end{aligned}$$

while uniformly in compact subsets of $\Omega_L^- \setminus (\overline{\Delta_\varepsilon(\zeta_1)} \cup \overline{\Delta_\varepsilon(\zeta_2)})$,

$$\begin{aligned} p_n(z) &= \left(\frac{G(z)}{\kappa}\right)^n \left([SN]_{11}(z) - e^{-2A\pi i n} H^{-2n}(z) [SN]_{12}(z)\right) \\ &= \left(\frac{G(z)}{\kappa}\right)^n \left(N_{11}(z) \left(1 + O\left(\frac{1}{n}\right)\right) \right. \\ &\qquad \left. - e^{-2A\pi i n} H^{-2n}(z) N_{12}(z) \left(1 + O\left(\frac{1}{n}\right)\right)\right). \end{aligned}$$

This finishes the proof of (2.19) and (2.20).

5.2. *Proof of Theorem 4*

Let $z \in \Delta_\varepsilon(\zeta_2)$; then by (4.21),

$$U(z) = S(z)P(z)K^{-1}(z),$$

where $K(z)$ is one of the matrices given in the right-hand side of (4.14). Gathering (4.19) and (4.20), we get that

$$P(z) = \sqrt{\pi} e^{\frac{\pi i}{6}} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \left(\frac{t_n^{1/4}}{a(z)} \right)^{\sigma_3} \Psi\left(t_n\right) e^{-A\pi i n \sigma_3} e^{n\phi(z)\sigma_3},$$

with $t_n \stackrel{\text{def}}{=} n^{2/3} f(z)$ and Ψ given by (4.18). For instance, if $z \in \Delta_\varepsilon(\zeta_2) \cap \Omega_R$, using (4.17) we get

$$\begin{aligned} U(z) = &\sqrt{\pi} e^{\frac{\pi i}{6}} S(z) \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \left(\frac{t_n^{1/4}}{a(z)} \right)^{\sigma_3} \Psi\left(n^{2/3} f(z)\right) \\ &\times e^{-A\pi i n \sigma_3} e^{n\phi(z)\sigma_3} \begin{pmatrix} 1 & 0 \\ d_n e^{2n\phi(z)} & 1 \end{pmatrix}. \end{aligned}$$

Observe that for $z \in \Delta_\varepsilon(\zeta_2) \cap \Omega_R^+$, we have $0 < \arg f(z) < 2\pi/3$, and we use the expression

$$\Psi(t) = \begin{pmatrix} \mathrm{Ai}(t) & \mathrm{Ai}(\omega^2 t) \\ \mathrm{Ai}'(t) & \omega^2 \mathrm{Ai}'(\omega^2 t) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}.$$

Hence,

$$\begin{pmatrix} U_{11}(z) \\ U_{21}(z) \end{pmatrix} = \sqrt{\pi}\, e^{n\phi(z)} S(z) \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \left(\frac{t_n^{1/4}}{a(z)} \right)^{\sigma_3} \begin{pmatrix} \mathcal{A}(t_n) \\ \mathcal{A}'(t_n) \end{pmatrix},$$

where

$$\mathcal{A}(t) \stackrel{\text{def}}{=} e^{-A\pi i n} \mathrm{Ai}(t) + 2i\, e^{\frac{\pi i}{3}} \sin(A\pi n) \mathrm{Ai}(\omega^2 t), \quad \omega = e^{2\pi i/3}.$$

Consequently,

$$U_{11}(z) = \sqrt{\pi}\, e^{n\phi(z)} \left(\frac{t_n^{1/4}}{a(z)} \mathcal{A}(t_n) \left(1 + O\left(\frac{1}{n}\right)\right) - \frac{a(z)}{t_n^{1/4}} \mathcal{A}'(t_n) \left(1 + O\left(\frac{1}{n}\right)\right)\right),$$

and taking into account (5.1) and the fact that in $\Delta_\varepsilon(\zeta_2) \cap \Omega_R^+$, $\exp(-\phi) = H$, we arrive at (2.23). Proceeding in a similar way, we see that this expression is also valid for z in the other regions of $\Delta_\varepsilon(\zeta_2)$.

5.3. *Proof of Theorem 5*

This theorem is a corollary of Theorems 1 and 2. First, taking into account (2.15) and that function N_{11}, defined in (2.13), has no zeros in the plane cut from ζ_1 to ζ_2, we see that zeros of $\{p_n\}$ cannot accumulate at $\mathbb{C} \setminus \mathrm{Pc}(\Sigma)$.

Consider in particular case C.3. Now the asymptotic location of the zeros of Jacobi polynomials depends also on the value

$$e^{-r} = \lim_{n\to\infty} |d_n|^{1/n} = \lim_{n\to\infty} |\sin(A\pi n)|^{1/n},$$

(assuming it exists), where d_n, defined in (4.2), depends upon the distance of $\alpha_n = An$ to the integers, in such a way that

$$e^{-r} = \lim_{n\to\infty} (\mathrm{dist}(\alpha_n, \mathbb{Z}))^{1/n}.$$

Let $z \in \mathrm{Int}(\mathrm{Pc}(\Sigma))$, that is, z lies in the bounded component limited by the contour Γ. We can choose $\varepsilon > 0$ small enough such that $z \notin \Delta_\varepsilon(\zeta_2)$. Then, the asymptotic formula (2.16),

$$\begin{aligned} p_n(z) = \frac{1}{\kappa^n} & \left(\left(G(z) w^2(z) \right)^{-n} N_{11}(z) \left(1 + O\left(\frac{1}{n}\right)\right) \right. \\ & \left. + 2i e^{-An\pi i} \sin(A\pi n)\, G^n(z) N_{12}(z) \left(1 + O\left(\frac{1}{n}\right)\right)\right), \end{aligned}$$

is valid. Hence, z is a zero of p_n only if

$$H^{-2n}(z) = -2i e^{-An\pi i} \sin(A\pi n) \frac{N_{12}(z)}{N_{11}(z)} \left(1 + O\left(\frac{1}{n}\right)\right).$$

Since N_{12}/N_{11} is uniformly bounded and uniformly bounded away from zero, we see that the zeros in this domain must satisfy

$$|H^{-1}(z)| = |\sin(A\pi n)|^{1/(2n)} \left(1 + O\left(\frac{1}{n}\right)\right) = e^{-r/2} \left(1 + O\left(\frac{1}{n}\right)\right).$$

It remains to use that $|H(z)| = e^{r/2}$ defines in $\mathrm{Int}(\mathrm{Pc}(\Sigma))$ the curve Γ_r.

Once we have established where the zeros accumulate, it remains to prove that they asymptotically distribute according to the corresponding measures in parts (i) and (ii). To this end we can use the second order linear differential equation satisfied by Jacobi polynomials $y_n = P_n^{(\alpha_n,\beta_n)}$ (see e.g. [35, §4.22]):

$$\begin{aligned} & (1 - z^2) y_n''(z) + \left[\beta_n - \alpha_n - (\alpha_n + \beta_n + 2)z\right] y_n'(z) \\ & \quad + n(n + \alpha_n + \beta_n + 1) y_n(z) = 0. \end{aligned}$$

If we rewrite it in terms of $h_n = y_n'/(n y_n)$, we obtain a Riccati differential equation:

$$\begin{aligned} & (1 - z^2)\left(\frac{1}{n} h_n'(z) + h_n^2(z)\right) + \frac{\beta_n - \alpha_n - (\alpha_n + \beta_n + 2)z}{n} h_n'(z) \\ & \quad + \frac{n + \alpha_n + \beta_n + 1}{n} = 0. \end{aligned} \tag{5.3}$$

Let ν_n denote the normalized zero counting measures of $y_n = P_n^{(\alpha_n,\beta_n)}$. By a weak compactness argument we know that there exists an infinite subsequence $\Lambda \subset \mathbb{N}$ and a unit measure ν such that $\nu_n \to \nu$, $n \in \Lambda$, in the weak*-topology. In the first part of this proof, we saw that $\mathrm{supp}(\nu)$ consists of a finite union of analytic arcs or curves, and every compact subset of $\mathbb{C} \setminus \mathrm{supp}(\nu)$ contains no zeros of $P_n^{(\alpha_n,\beta_n)}$ for n sufficiently large.

Hence,

$$h_n(z) = \int \frac{d\nu_n(t)}{z-t} \longrightarrow h(z) = \int \frac{d\nu(t)}{z-t}, \quad n \in \Lambda,$$

locally uniformly in $\mathbb{C} \setminus \operatorname{supp}(\nu)$. Taking limits in (5.3) we obtain that h satisfies the quadratic equation

$$(1-z^2)\, h^2(z) + [B - A - (A+B)z]\, h(z) + A + B + 1 = 0,$$

so that

$$\int \frac{d\nu(t)}{z-t} = \frac{A+B+2}{2} \, \frac{R(z)}{z^2-1} - \frac{1}{2}\left(\frac{A}{z-1} + \frac{B}{z+1}\right), \quad z \in \mathbb{C} \setminus \operatorname{supp}(\nu).$$

By Sokhotsky-Plemelj's formulas, on every arc of $\operatorname{supp}(\nu)$,

$$d\nu(z) = \frac{A+B+2}{2\pi i} \, \frac{R_+(z)}{z^2-1} \, dz \tag{5.4}$$

(this derivation might serve as a motivation of definition (2.1) of the branch points $\zeta_{1,2}$).

Now, we are concerned with proving part (ii) of the theorem, related to case C.3. First, consider the generic case when $r = 0$. In this case, the measure μ in (2.5) is supported on $\Sigma = \Gamma \cup [\zeta_1, \zeta_2]$. Thus, by (5.4), $\mu' = \nu'$ a.e. on $\operatorname{supp}(\nu)$, μ, ν being unit measures. Therefore, $\nu = \mu$. The proof in the case $0 < r < \infty$ is similar, but with measures μ_r, given in (2.24), in place of μ. Finally, for the degenerate case $r = \infty$, which takes place when parameters α_n approach the integers faster than exponentially, it is enough to take into account that the Cauchy transform of the measure $d\sigma = -A\delta_1$ is $\widehat{\sigma}(z) = -A/(z-1)$.

Finally, the proof for case C.2 is totally analogous.

Acknowledgments

The research of A.M.F. was supported, in part, by a research grant from the Ministry of Science and Technology (MCYT) of Spain, project code BFM2001-3878-C02, by NATO Collaborative Linkage Grant “Orthogonal Polynomials: Theory, Applications and Generalizations”, ref. PST.CLG.979738, and by Research Network “Network on Constructive Complex Approximation (NeCCA)”, INTAS 03-51-6637.

The research of R.O. was partially supported by grants from Spanish MCYT (BFM2001-3411) and Gobierno Autónomo de Canarias (PI2002/136).

Finally, the authors are grateful to Prof. A. B. J. Kuijlaars for stimulating discussions on the Riemann–Hilbert analysis.

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